

Parabolically induced unitary representations of the universal group $U(F)^+$ are C_0

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September 8, 2014

Abstract

By employing a new strategy we prove that all parabolically induced unitary representations of the Burger–Mozes universal group $U(F)^+$, with F being primitive, have all their matrix coefficients vanishing at infinity. This generalizes the same well-known result for the universal group $U(F)^+$, when F is 2-transitive.

1 Introduction

The motivation of this paper is coming from the following question proposed by Burger and Mozes, and Caprace:

Question 1.1. *Let G be a locally compact, topologically simple group acting continuously and properly by type-preserving automorphisms on a d -regular tree T_d . If G admits the Howe–Moore property is it true that G acts 2-transitively on the boundary of the tree T_d ?*

Recall that a locally compact group G enjoys the Howe–Moore property if all matrix coefficients of all the group unitary representations that are without non-zero G -invariant vectors vanish at infinity. This property is well-known to hold for connected, non-compact, simple real Lie groups, with finite center, and for their totally disconnected analogs, namely, isotropic simple algebraic groups over non Archimedean local fields and closed, topologically simple subgroups of $\text{Aut}(T)$ that act 2-transitively on the boundary ∂T , where T is a bi-regular with valence ≥ 3 at every vertex. In Ciobotaru [Cio15] a unified proof is given for all these examples. Apart for those above mentioned groups, there is no other known example of a locally compact group admitting the Howe–Moore property. It is therefore legitimate to ask if the Howe–Moore property holds only for those mentioned simple algebraic groups over local fields and for their relatives coming from groups acting on trees. As this question is more difficult to be studied in full generality, we simply restrict to Question 1.1, where things should be more easy. To

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make things even more easier, a test case is proposed to be studied with respect to Question 1.1: the universal group $U(F)$ introduced by Burger and Mozes [BM00, Section 3]. In fact, as $U(F)$ is not a simple group, we study its simple subgroup $U(F)^+$ (for its definition and its properties see Section 2). The main reason why the universal group $U(F)^+$ is suggested as the first example that should be studied towards answering Question 1.1 is the following. By Caprace and De Medts [CDM11, Prop. 4.1], we know that $F < \text{Sym}(\{1, \dots, d\})$ is primitive if and only if every proper open subgroup of $U(F)^+$ is compact. The latter condition that every proper open subgroup is compact is well-known to be satisfied by any group that enjoys the Howe–Moore property; however, it is not sufficient, in general, to imply the Howe–Moore property. To conclude, we emphasize a second question that stems from Question 1.1:

Question 1.2. *Is it true that the group $U(F)^+$ does not have the Howe–Moore property if F is primitive but not 2-transitive?*

Although we have reduced Question 1.1 to the very concrete Question 1.2, the perspective is not so easy. The reason is that the theory of unitary representations for closed subgroups of $\text{Aut}(T_d)$ that do not act 2-transitively on the boundary of T_d is very less developed, even in the particular case of the universal group $U(F)^+$, with F being primitive but not 2-transitive. This is also the reason why this paper treats only parabolically induced unitary representations of the universal group $U(F)^+$, with F being primitive. Unfortunately for our Questions 1.1 and 1.2, the result of this paper cannot give any answer, the questions still remaining open. However, we estimate that the techniques introduced in this paper can serve to prove other results in that direction.

The goal of this paper is to give a proof of the following vanishing result:

Theorem 1.3. *(See Theorem 4.13) Let F be primitive and let $\xi \in \partial T_d$. Let H be a closed subgroup of $U(F)_\xi^+$. Let (σ, \mathcal{K}) be a unitary representation of H and consider on $U(F)^+/H$ a $U(F)^+$ -quasi-invariant regular Borel measure μ_ρ given by the rho-function $\rho : U(F)^+ \rightarrow \mathbb{R}_+^*$.*

Then the induced unitary representation $(\pi_{\sigma, \mu_\rho}, \mathcal{H}_{\mu_\rho})$ of $U(F)^+$ has all its matrix coefficients vanishing at infinity.

As a consequence of the result of Burger and Mozes [BM00] we know that Theorem 1.3 is true when F is 2-transitive; however, the techniques used in Burger and Mozes [BM00] (see also Ciobotaru [Cio15]) cannot be used to prove Theorem 1.3 when F is primitive but not 2-transitive. The main reason is that for the former case the group $U(F)^+$ admits a KA^+K decomposition with A^+ being an abelian sub semi-group of $U(F)^+$. This gives us the right to use the theory of normal operators (see Ciobotaru [Cio15, Theorem 1.2]). When F is primitive but not 2-transitive, the sub semi-group A^+ of $U(F)^+$, coming from the polar-like decomposition, is not anymore abelian, therefore a different strategy is needed.

The main steps to prove Theorem 1.3 are the following. From the results proved in Section 3, to evaluate the matrix coefficients of the induced unitary representation

$(\pi_{\sigma, \mu_\rho}, \mathcal{H}_{\mu_\rho})$ of $U(F)^+$, it is enough to evaluate integrals of the form

$$\int_{t_n(f_1KH) \cap f_2KH} \left(\frac{\rho(t_n^{-1}x)}{\rho(x)} \right)^{1/2} d\mu_\rho(xH),$$

where $f_1, f_2 \in U(F)^+$ are considered to be fixed, $\{t_n\}_{n>0} \subset U(F)^+$ is such that $t_n \rightarrow \infty$ and $K := U(F)_x^+$, with x a vertex of T_d . From here it remains to evaluate the intersection $g(f_1KH) \cap f_2KH$, for $g \in U(F)^+$, and to bound from above the integrant $\left(\frac{\rho(g^{-1}(x))}{\rho(x)} \right)^{1/2}$, where $x \in g(f_1KH) \cap f_2KH$.

For $x \in g(f_1KH) \cap f_2KH$ there exist $k, k' \in K$ and $h, h' \in H$, such that $x = gf_1kh = f_2k'h'$. Therefore we have that

$$\left(\frac{\rho(g^{-1}x)}{\rho(x)} \right)^{1/2} = \left(\frac{\rho(f_1kh)}{\rho(f_2k'h')} \right)^{1/2} = \left(\frac{\rho(f_1k)\Delta_H(h)}{\rho(f_2k')\Delta_H(h')} \right)^{1/2} < C \left(\frac{\Delta_H(h)}{\Delta_H(h')} \right)^{1/2},$$

where the constant C depends only on K, ρ and f_1, f_2 .

We distinguish two cases: when H does not contain hyperbolic elements (in this case H is unimodular) and when H does contain hyperbolic elements (therefore, H is not unimodular anymore). In the case when H does not contain hyperbolic elements it is enough to evaluate $\mu_\rho(t_n(f_1KH) \cap f_2KH)$, where $t_n \rightarrow \infty$. This is explained in Section 4.1 and in the proof of Theorem 4.4.

When H does contain hyperbolic elements, we cannot bound from above $\left(\frac{\Delta_H(h)}{\Delta_H(h')} \right)^{1/2}$, simply because the modular function of H is not anymore the constant function $\mathbb{1}_H$. In this case we proceed as follows. As above, there is a constant $C > 0$, depending only on K, ρ and f_1, f_2 such that

$$\int_{t_n f_1KH \cap f_2KH} \left(\frac{\rho(t_n^{-1}x)}{\rho(x)} \right)^{1/2} d\mu_\rho(xH) \leq C \int_{\sqcup_{i \in I_n} f_2 k_{i,n} H} \Delta_H(h_{i,n})^{-1/2} d\mu_\rho(f_2 k_{i,n} H),$$

where $I_n, k_{i,n}, h_{i,n}$ depend on t_n and they are such that $t_n f_1 k_{i,n} = f_2 k_{i,n} h_{i,n} \in f_2 k_{i,n} H$ (see Lemma 4.7).

It remains to bound from above the integral $\int_{\sqcup_{i \in I_n} f_2 k_{i,n} H} \Delta_H(h_{i,n})^{-1/2} d\mu_\rho(f_2 k_{i,n} H)$.

To do this, it is natural to evaluate the set of all right cosets $[h_{i,n}] \in H/(H \cap (U(F)^+)_\xi^0)$, for $i \in I_n$, and for each such right coset $[h]$ to evaluate $\mu_\rho(K_{[h]})\Delta_H(h)^{-1/2}$, where $K_{[h]} \subset K$ is the domain of all solutions k_1 of the equation $k_1 f_2^{-1} t_n f_1 k_2 = h$, with $k_1, k_2 \in K/(K \cap H)$ and $h \in [h]$, where t_n is fixed. Next, by ‘summing up’ all $\mu_\rho(K_{[h]})\Delta_H(h)^{-1/2}$ corresponding to t_n , we obtain a sequence that tends to zero, when $t_n \rightarrow \infty$. This is explained in Section 4.2 and in the proof of Theorem 4.12.

The paper is structured as follows. Section 2 recalls the definitions and gathers some properties of the Burger–Mozes universal group $U(F)^+$ that are used in the sequel. In

Section 3 the general theory of induced unitary representations of locally compact groups is presented together with general tools that are used to prove Theorem 1.3. Finally, the main steps that were resumed above are implemented in Section 4, where Theorem 1.3 is also proved.

2 On the universal group $U(F)^+$

As we mentioned in the Introduction, the group $U(F)$ was introduced by Burger–Mozes in [BM00, Section 3]. In his PhD thesis [Ama03], Amann studies this group from the point of view of its unitary representations.

This section is meant to introduce the definition and to gather some well-known properties of the Burger–Mozes universal group $U(F)$ that are used in this paper. For a more complete list of (old and new) properties, the reader can consult Ciobotaru [Cio14, Section 2.2].

First, let us fix some notation. Denote by \mathcal{T} a **d -regular tree, with $d \geq 3$** , and by $\text{Aut}(\mathcal{T})$ its full group of automorphisms.

Definition 2.1. Let $i : E(\mathcal{T}) \rightarrow \{1, \dots, d\}$ be a function from the set $E(\mathcal{T})$ of unoriented edges of the tree \mathcal{T} such that its restriction to the star $E(x)$ of every vertex $x \in \mathcal{T}$ is in bijection with $\{1, \dots, d\}$. A function i with those properties is called a **legal coloring** of the tree \mathcal{T} .

Definition 2.2. Let F be a subgroup of permutations of the set $\{1, \dots, d\}$ and let i be a legal coloring of \mathcal{T} . We define the **universal group**, with respect to F and i , to be

$$U(F) := \{g \in \text{Aut}(\mathcal{T}) \mid i \circ g \circ (i|_{E(x)})^{-1} \in F, \text{ for every } x \in \mathcal{T}\}.$$

By $U(F)^+$ we denote the subgroup generated by the edge-stabilizing elements of $U(F)$. Moreover, Proposition 52 of Amann [Ama03] tells us that the group $U(F)$ is independent of the legal coloring i of \mathcal{T} .

Immediately from the definition we deduce that $U(F)$ and $U(F)^+$ are closed subgroups of $\text{Aut}(\mathcal{T})$. Notice that, when F is the full permutation group $\text{Sym}(\{1, \dots, d\})$, $U(F) = \text{Aut}(\mathcal{T})$ and $U(F)^+$ is simply denoted by $\text{Aut}(\mathcal{T})^+$, which is an index 2, simple subgroup of $\text{Aut}(\mathcal{T})$ (for the latter assertion see Tits [Tit70]).

Before stating some main properties of the universal group, we record another definition which is given in general, for a locally finite tree, and which is used in the sequel as a key property.

Definition 2.3 (See Tits [Tit70]). Let T be a locally finite tree and let $G \leq \text{Aut}(T)$ be a closed subgroup. We say that G has **Tits' independence property** if for every edge e of T we have the equality $G_e = G_{T_1} G_{T_2}$, where T_i are the two infinite half sub-trees of T emanating from the edge e and G_{T_i} is the pointwise stabilizer of the half-tree T_i .

We mention that Tits' independence property guarantees the existence of 'enough' rotations in the group G . It is used in the work of Tits as a sufficient condition to prove simplicity of 'large' subgroups of $\text{Aut}(\mathcal{T})$ (see Tits [Tit70]). In his thesis, [Ama03, Theorem 2], Amann employs it to give a complete classification of all super-cuspidal representations of a closed subgroup in $\text{Aut}(\mathcal{T})$ acting transitively on the vertices and on the boundary of \mathcal{T} and having Tits' independence property. For a closed subgroup G of $\text{Aut}(\mathcal{T})$ that acts transitively on the vertices and on the boundary of \mathcal{T} but which does not enjoy Tits' independence property less is known about the complete classification of all its super-cuspidal representations. In contrast, for the above mentioned groups, with or without Tits' independence property, the remaining two classes of irreducible unitary representations, namely the special and respectively, the spherical ones, are completely classified in Figà-Talamanca–Nebbia [FTN91, Chapter III, resp. Chapter II].

Following Burger–Mozes and Amann [BM00, Ama03], we enumerate the following properties of the groups $U(F)$ and $U(F)^+$:

- 1) $U(F)$ and $U(F)^+$ have Tits' independence property;
- 2) $U(F)^+$ is trivial or simple;
- 3) if F acts transitively on the set $\{1, \dots, d\}$ then $U(F)$ acts transitively on the vertices of \mathcal{T} and it is a unimodular group;
- 4) $U(F)$ and $U(F)^+$ act 2-transitively on the boundary $\partial\mathcal{T}$ if and only if F is 2-transitive;
- 5) When F is transitive and generated by its point stabilizers, by Amann [Ama03, Prop. 57], we have that $U(F)^+$ is edge-transitive, $U(F)^+ = U(F) \cap \text{Aut}(\mathcal{T})^+$ and $U(F)^+$ is of index 2 in $U(F)$. For example, this is the case when F is primitive but not cyclic of prime order. When F is primitive and not generated by its point stabilizers, all point stabilizers of F are equal. This implies that all point stabilizers of F are just the identity, that F is cyclic of prime order and that the group $U(F)^+$ is trivial. Under these hypotheses, we obtain that $U(F)$ is a nontrivial, discrete subgroup of $\text{Aut}(\mathcal{T})$. In order to avoid heavy formulation, we simply use ' F is primitive' to mean that ' F is primitive but not cyclic of prime order'.

As explained in the Introduction, the following proposition states an important property regarding proper open subgroups of $U(F)^+$. This property is a useful tool employed in some the proofs of the results obtained in the next sections.

Proposition 2.4. *(See Caprace–De Medts [CDM11, Prop. 4.1]) The subgroup F is primitive if and only if every proper open subgroup of $U(F)^+$ is compact.*

To avoid heavy notation, for the rest of the article we refer to the following convention.

Convention 2.5. Let F be primitive. Consider fixed a coloring ι of \mathcal{T} , a vertex $x \in \mathcal{T}$ and an edge e of the star of x . For simplicity, set $\mathbb{G} := U(F)^+$ and $K := U(F)_x^+$. Let $\mathcal{T}_{x,e}$

be the half-tree of \mathcal{T} that emanates from the vertex x and that contains the edge e . For every two points $y, z \in \mathcal{T} \cup \partial\mathcal{T}$, we denote by $[y, z]$ the unique geodesic between y and z in $\mathcal{T} \cup \partial\mathcal{T}$. For a hyperbolic element γ in \mathbb{G} , we denote $|\gamma| := \min_{x \in \mathcal{T}} \{\text{dist}_{\mathcal{T}}(x, \gamma(x))\}$, which is called the translation length of γ , and set $\text{Min}(\gamma) := \{x \in \mathcal{T} \mid \text{dist}_{\mathcal{T}}(x, \gamma(x)) = |\gamma|\}$.

When F is primitive but not 2-transitive, the universal group \mathbb{G} still enjoys some of the properties of closed, non-compact subgroups of $\text{Aut}(\mathcal{T})$ that act 2-transitively on the boundary $\partial\mathcal{T}$.

Remark 2.6. As F is primitive, given an edge $e' \in V(\mathcal{T})$ at odd distance from e , one can construct a hyperbolic element in \mathbb{G} translating e to e' . Moreover, every hyperbolic element in \mathbb{G} has even translation length.

Lemma 2.7. (*KA^+K decomposition*) *Let F be primitive. Then \mathbb{G} admits a KA^+K decomposition, where*

$$A^+ := \{\gamma \in \mathbb{G} \mid \gamma \text{ is hyperbolic translating the edge } e \text{ inside } \mathcal{T}_{x,e}\} \cup \{\text{id}\}.$$

Proof. Let $g \in \mathbb{G}$. If $g(x) = x$, then $g \in K$. If not, then $g(x) \neq x$. Consider the geodesic segment $[x, g(x)]$ in \mathcal{T} and denote by e_1 the edge of the star of x that belongs to $[x, g(x)]$. Notice that $[x, g(x)]$ has even length and that there exists $k \in K$ such that $k(e_1) = e$; therefore, $kg(x) \in \mathcal{T}_{x,e}$. Then, by Remark 2.6, there is a hyperbolic element $\gamma \in \mathbb{G}$ of translation length equal to the length of $[x, g(x)]$, that translates the edge e inside $\mathcal{T}_{x,e}$ and such that $\gamma(x) = kg(x)$; thus $\gamma^{-1}kg \in K$. Notice that the KA^+K decomposition of an element $g \in \mathbb{G}$ is not unique. \square

3 Induced unitary representations

In the context of the unitary representation theory, a powerful general technique to obtain new examples of unitary representations of a locally compact group is to start with a known unitary representation of a closed subgroup and then to extend it on the bigger group. This is called induction.

For example, some irreducible unitary representations of a real reductive Lie group can be obtained by parabolic induction, where the word ‘parabolic’ refers to its parabolic subgroups. For the reader convenience, in this section we recall the general theory of induced unitary representations of locally compact groups with respect to their closed subgroups. We also give general tools that are used in Section 4 where Theorem 1.3 is proved.

The general setting of induced unitary representations is the following (see Bekka–de la Harpe–Valette [BdlHV08, Appendices B and E]): Let G be a locally compact group, H a closed subgroup of G and (σ, \mathcal{K}) a unitary representation of H . We want to extend (σ, \mathcal{K}) to a unitary representation of G .

First of all, endow G/H with the quotient topology, meaning that the canonical projection $p : G \rightarrow G/H$ is continuous and open. In order to construct a Hilbert space

corresponding to G and which is ‘induced’ from the unitary representation (σ, \mathcal{K}) , we need a measure on G/H . By convention, all Haar measures used in this paper are considered to be left invariant.

Definition 3.1. (See Bekka–de la Harpe–Valette [BdlHV08, Appendix B]) A **rho-function** on G is a continuous function $\rho : G \rightarrow \mathbf{R}_+^*$ satisfying the equality

$$\rho(xh) = \frac{\Delta_H(h)}{\Delta_G(h)} \rho(x) \text{ for all } x \in G, h \in H, \quad (1)$$

where Δ_G, Δ_H are the modular functions on G , respectively on H .

By Bekka–de la Harpe–Valette [BdlHV08, Thm. B.14], we have that

- i) for a given rho-function on G , there exists a canonical G –quasi-invariant regular Borel measure μ_ρ on G/H ;
- ii) conversely, only ‘strongly’ G –quasi-invariant regular Borel measures on G/H come from a continuous rho-function on G , where ‘strongly’ means that the corresponding Radon–Nikodym 1-cocycle is a continuous map.

Corresponding to the unitary representation (σ, \mathcal{K}) of H , let \mathcal{A} be the space of all mappings $\xi : G \rightarrow \mathcal{K}$ with the following properties:

- i) ξ is continuous;
- ii) $p(\text{supp } \xi)$ is compact;
- iii) $\xi(xh) = \sigma(h^{-1})\xi(x)$ for all $x \in G$ and all $h \in H$.

Examples of such mapping are coming from the following construction. Let $C_c(G)$ be the space of all continuous, complex valued functions on G with compact support. For $f \in C_c(G)$ and $v \in \mathcal{K}$, let $\xi_{f,v} : G \rightarrow \mathcal{K}$ be the mapping given by

$$x \mapsto \xi_{f,v}(x) := \int_H f(xh)\sigma(h)(v)dh,$$

where dh is the Haar measure on H and $\int_H f(xh)\sigma(h)(v)dh$ represents the unique element in \mathcal{K} such that $\left\langle \int_H f(xh)\sigma(h)(v)dh, w \right\rangle_{\mathcal{K}} = \int_H \langle f(xh)\sigma(h)(v), w \rangle_{\mathcal{K}} dh$, for all $w \in \mathcal{K}$. Here $\langle, \rangle_{\mathcal{K}}$ represents the inner product on the Hilbert space \mathcal{K} . By Bekka–de la Harpe–Valette [BdlHV08, Prop. E.1.1 and Lem. E.1.3] the mapping $\xi_{f,v}$ belongs to \mathcal{A} . Moreover, the linear span of $\{\xi_{f,v} \mid f \in C_c(G), v \in \mathcal{K}\}$ is dense in \mathcal{A} with respect to the norm induced from the following inner product.

Suppose that G/H is endowed with a G -quasi-invariant regular Borel measure μ . By Bekka–de la Harpe–Valette [BdlHV08, Appendix E] the space \mathcal{A} admits a positive definite hermitian form given by

$$\langle \xi, \eta \rangle := \int_{G/H} \langle \xi(x), \eta(x) \rangle_{\mathcal{K}} d\mu(xH),$$

for every $\xi, \eta \in \mathcal{A}$. Denote by $\mathcal{H}_{\sigma, \mu}$ the Hilbert space completion of \mathcal{A} with respect to \langle, \rangle .

Definition 3.2. Let G be a locally compact group, H a closed subgroup of G and (σ, \mathcal{K}) a unitary representation of H . Suppose that G/H is endowed with a G -quasi-invariant regular Borel measure μ_ρ determined by a rho-function ρ on G . With respect to this data, the **induced unitary representation** $(\pi_{\sigma, \mu_\rho}, \mathcal{H}_{\sigma, \mu_\rho})$ of G is defined as follows.

For every $g \in G$, we define the unitary operator $\pi_{\sigma, \mu_\rho}(g)$ on \mathcal{A} by

$$\pi_{\sigma, \mu_\rho}(g)(\xi)(x) := \left(\frac{\rho(g^{-1}x)}{\rho(x)} \right)^{1/2} \xi(g^{-1}x),$$

where $\xi \in \mathcal{A}$ and $x \in G$. Extend this unitary operator to the Hilbert space $\mathcal{H}_{\sigma, \mu_\rho}$.

By Bekka–de la Harpe–Valette [BdlHV08, Prop. E.1.4], this is indeed a unitary representation of G on the Hilbert space $\mathcal{H}_{\sigma, \mu_\rho}$.

The following lemmas are useful tools for establishing vanishing results for the matrix coefficients of induced unitary representations.

Lemma 3.3. *Let G be a locally compact group. Let H and $K < G$ be a closed, respectively, a compact subgroup. Consider on G/H a G -quasi-invariant regular Borel measure μ_ρ given by the rho-function $\rho : G \rightarrow \mathbb{R}_+^*$. Then to μ_ρ it is associated, in a canonical way, a G -quasi-invariant regular Borel measure μ'_ρ on G/H which is left K -invariant.*

Proof. Let $\rho' : G \rightarrow \mathbb{R}_+^*$ be the function defined by $g \in G \mapsto \rho'(g) := \int_K \rho(kg) dk$, where dk is the Haar measure on K . It is easy to see that ρ' is continuous and that it satisfies the equation (1) from Definition 3.1. We obtain that ρ' is a rho-function and in addition it is left K -invariant. Let $\mu_{\rho'}$ be the G -quasi-invariant regular Borel measure on G/H associated to ρ' . As the Radon–Nikodym derivative of $\mu_{\rho'}$ is $\frac{d\mu_{\rho'}}{d\mu_\rho}(xH) = \frac{\rho'(yx)}{\rho'(x)}$, for every $x, y \in G$, we obtain the left K -invariance of the measure $\mu_{\rho'}$. \square

Lemma 3.4. *Let G be a locally compact group. Let H and $K < G$ be a closed, respectively, a compact subgroup, such that $K' := H \cap K$ has infinite index in K . Consider on G/H a G -quasi-invariant regular Borel measure μ_ρ which is given by the rho-function $\rho : G \rightarrow \mathbb{R}_+^*$ and which is left K -invariant. Assume that $\mu_\rho(KH) \neq 0$.*

Then the index of K' in K is uncountable. In particular, $\mu_\rho(H) = 0$.

Proof. By Lemma 3.3, we know that there exists on G/H a G -quasi-invariant regular Borel measure μ_ρ which is left K -invariant.

By the definition of a regular Borel measure, we have that $\mu_\rho(KH) < \infty$. Suppose that the index of K' in K is countable; therefore, there exist $\{k_n\}_{n \in \mathbb{N}} \subset K \setminus K'$ such that $K = \bigsqcup_n k_n K'$. We have that $KH = \bigsqcup_n k_n H$ and $\mu_\rho(KH) = \sum_{n \in \mathbb{N}} \mu_\rho(k_n H) = \sum_{n \in \mathbb{N}} \mu_\rho(H)$, as μ_ρ is countably additive and K -invariant. Because $\mu_\rho(KH) < \infty$ we conclude that $\mu_\rho(H)$ must be zero and so $\mu_\rho(KH)$ is zero too, which contradicts the hypothesis. Therefore, the index of K' in K must be uncountable. \square

Remark 3.5. (See Bekka–de la Harpe–Valette, Proposition E.1.5 of [BdlHV08]) Consider two G -quasi-invariant regular Borel measures $\mu_{\rho_1}, \mu_{\rho_2}$ on G/H , that correspond, respectively, to two rho-functions ρ_1 and ρ_2 on G . Then, for any unitary representation (σ, \mathcal{K}) of H , the corresponding induced unitary representations $(\pi_{\sigma, \mu_{\rho_1}}, \mathcal{H}_{\mu_{\rho_1}}), (\pi_{\sigma, \mu_{\rho_2}}, \mathcal{H}_{\mu_{\rho_2}})$ of G are equivalent. This implies, for example, that if $(\pi_{\sigma, \mu_{\rho_1}}, \mathcal{H}_{\mu_{\rho_1}})$ has all its matrix coefficients vanishing at infinity then the same is true for $(\pi_{\sigma, \mu_{\rho_2}}, \mathcal{H}_{\mu_{\rho_2}})$.

Remark 3.6. Let G be a locally compact group and H be a closed subgroup of G . Suppose that G/H is endowed with a G -quasi-invariant regular Borel measure μ_ρ determined by a rho-function ρ on G . Let K be a compact subset of G . Then $g\mu_\rho(KH) = \mu_\rho(g^{-1}KH) = \int_{G/H} \mathbb{1}_{KH}(x) dg \mu_\rho(x) = \int_{KH} \frac{\rho(gx)}{\rho(x)} d\mu_\rho(x) \leq C \int_{KH} \mathbb{1}_{KH}(x) d\mu_\rho(x) = C\mu_\rho(KH)$,

where $C \geq \max_{x \in K} \left\{ \frac{\rho(gx)}{\rho(x)} \right\}$.

Lemma 3.7. Let G be a locally compact group and H be a closed subgroup of G . Let (σ, \mathcal{K}) be a unitary representation of H and consider on G/H a G -quasi-invariant regular Borel measure μ_ρ given by the rho-function $\rho : G \rightarrow \mathbb{R}_+^*$. Assume there exist a sequence $\{t_k\}_{k>0}$ of G and $\eta_1, \eta_2 \in \mathcal{H}_{\mu_\rho}$ such that $t_k \rightarrow \infty$ and $|\langle \pi_{\sigma, \mu_\rho}(t_k)\eta_1, \eta_2 \rangle| \rightarrow 0$.

Then there exist $\eta'_1, \eta'_2 \in \text{span}(\{\xi_{f,v} \mid f \in C_c(G), v \in \mathcal{K}\})$, $\delta > 0$ and a subsequence $\{t_{k_m}\}_{k_m>0}$, with $t_{k_m} \rightarrow \infty$, such that $|\langle \pi_{\sigma, \mu_\rho}(t_{k_m})\eta'_1, \eta'_2 \rangle| > \delta$, for every k_m .

Proof. As, by hypothesis, $|\langle \pi_{\sigma, \mu_\rho}(t_k)\eta_1, \eta_2 \rangle| \rightarrow 0$, there exist a subsequence $\{t_{k_m}\}_{k_m>0}$, with $t_{k_m} \rightarrow \infty$, $N \in \mathbb{N}$ and $\delta' > 0$ such that $|\langle \pi_{\sigma, \mu_\rho}(t_{k_m})\eta_1, \eta_2 \rangle| > \delta'$, for every $k_m \geq N$.

By construction, the linear span of $\{\xi_{f,v} \mid f \in C_c(G), v \in \mathcal{K}\}$ is dense in \mathcal{A} and therefore, also in \mathcal{H}_{μ_ρ} . Let $\{\eta_{1,n}\}_{n>0}, \{\eta_{2,n}\}_{n>0} \subset \text{span}(\{\xi_{f,v} \mid f \in C_c(G), v \in \mathcal{K}\})$ be two sequences tending to η_1 and respectively, η_2 in the norm of \mathcal{H}_{μ_ρ} . We obtain the following inequality:

$$|\langle \pi_{\sigma, \mu_\rho}(t)(\eta_1 - \eta_{1,n}), \eta_2 \rangle| \leq \|\eta_1 - \eta_{1,n}\|_{\mathcal{H}_{\mu_\rho}} \cdot \|\eta_2\|_{\mathcal{H}_{\mu_\rho}},$$

for every $n \in \mathbb{N}$ and every $t \in G$.

From our assumption one has:

$$\begin{aligned} \delta' &< |\langle \pi_{\sigma, \mu_\rho}(t_{k_m})\eta_1, \eta_2 \rangle| \leq |\langle \pi_{\sigma, \mu_\rho}(t_{k_m})(\eta_1 - \eta_{1,n}), \eta_2 \rangle| \\ &\quad + |\langle \pi_{\sigma, \mu_\rho}(t_{k_m})\eta_{1,n}, \eta_2 \rangle| \\ &\leq \|\eta_1 - \eta_{1,n}\|_{\mathcal{H}_{\mu_\rho}} \cdot \|\eta_2\|_{\mathcal{H}_{\mu_\rho}} \\ &\quad + |\langle \pi_{\sigma, \mu_\rho}(t_{k_m})\eta_{1,n}, \eta_2 \rangle|, \end{aligned}$$

for every $n \in \mathbb{N}$ and $k_m \geq N$.

By choosing an n and $\eta_{1,n}$ such that $\|\eta_1 - \eta_{1,n}\|_{\mathcal{H}_{\mu_\rho}} \cdot \|\eta_2\|_{\mathcal{H}_{\mu_\rho}} < \varepsilon$, with ε very small, we have $|\langle \pi_{\sigma,\rho}(t_{k_m})\eta_{1,n}, \eta_2 \rangle| > \delta' - \varepsilon$ for every $k_m \geq N$. By proceeding in the same way with the vector η_2 , we obtain an n' and $\eta_{2,n'}$ such that $|\langle \pi_{\sigma,\rho}(t_{k_m})\eta_{1,n}, \eta_{2,n'} \rangle| > \delta' - 2\varepsilon > 0$, for every $k_m \geq N$ and a very small $\varepsilon > 0$.

By taking $\delta := \delta' - 2\varepsilon$, $\eta'_1 := \eta_{1,n}$, $\eta'_2 := \eta_{2,n'}$ and the sequence $\{t_{k_m}\}_{k_m > 0}$, the lemma follows. \square

Lemma 3.8. *Let G be a locally compact group, H a closed subgroup of G and K a compact-open neighborhood in G of the identity. Let (σ, \mathcal{K}) be a unitary representation of H and consider on G/H a G -quasi-invariant regular Borel measure μ_ρ given by the rho-function $\rho : G \rightarrow \mathbb{R}_+^*$. Let $\eta_1, \eta_2 \in \text{span}(\{\xi_{f,v} \mid f \in C_c(G), v \in \mathcal{K}\})$.*

Then there exist a constant $C > 0$, $N_1, N_2 \in \mathbb{N}$ and elements $\{h_i\}_{i \in \{1, \dots, N_1\}}, \{h'_j\}_{j \in \{1, \dots, N_2\}} \subset G$, all of them depending only on η_1 and η_2 , such that

$$\begin{aligned} |\langle \pi_{\sigma,\mu_\rho}(t)\eta_1, \eta_2 \rangle| &= |\langle \eta_1, \pi_{\sigma,\mu_\rho}(t^{-1})\eta_2 \rangle| \\ &\leq \sum_{i,j=1}^{N_1, N_2} \int_{t(h_i KH) \cap h'_j KH} \left| \left(\frac{\rho(t^{-1}x)}{\rho(x)} \right)^{1/2} \langle \eta_1(t^{-1}x), \eta_2(x) \rangle_{\mathcal{K}} d\mu_\rho \right| \\ &\leq C \sum_{i,j=1}^{N_1, N_2} \int_{t(h_i KH) \cap h'_j KH} \left(\frac{\rho(t^{-1}x)}{\rho(x)} \right)^{1/2} d\mu_\rho, \end{aligned} \quad (2)$$

for every $t \in G$.

Moreover, we have that

$$\begin{aligned} \int_{t(h_i KH) \cap h'_j KH} \left| \left(\frac{\rho(t^{-1}x)}{\rho(x)} \right)^{1/2} \langle \eta_1(t^{-1}x), \eta_2(x) \rangle_{\mathcal{K}} d\mu_\rho(x) \right| &= \\ \int_{h_i KH \cap t^{-1}(h'_j KH)} \left| \left(\frac{\rho(ty)}{\rho(y)} \right)^{1/2} \langle \eta_1(y), \eta_2(ty) \rangle_{\mathcal{K}} d\mu_\rho(y) \right|. \end{aligned} \quad (3)$$

Proof. Notice that the last assertion of the lemma follows using the change of variables $y := t^{-1}x$ and the fact that the function ρ is positive.

Let $t \in G$. As $\eta_1, \eta_2 \in \text{span}(\{\xi_{f,v} \mid f \in C_c(G), v \in \mathcal{K}\})$, they depend on a finite number of functions from $C_c(G)$. Denote by $A, B \subset G$ the union of the support of those functions that define η_1 and respectively, η_2 . A and B are compact subsets of G . Cover A and respectively, B , with open sets of the form hK , where $h \in A$ and respectively, $h \in B$. From these open covers extract finite ones that cover A and respectively, B . By making a choice and fixing the notations, consider that $A \subset \bigcup_{i=1}^{N_1} h_i K$ and $B \subset \bigcup_{j=1}^{N_2} h'_j K$, where $h_i, h'_j \in G$ and $N_1, N_2 \in \mathbb{N}$.

We obtain:

$$\begin{aligned}
& |\langle \pi_{\sigma, \mu_\rho}(t) \eta_1, \eta_2 \rangle| \\
&= \left| \int_{G/H} \left(\frac{\rho(t^{-1}x)}{\rho(x)} \right)^{1/2} \langle \eta_1(t^{-1}x), \eta_2(x) \rangle_{\mathcal{K}} d\mu_\rho(xH) \right| \\
&\leq \int_{t(AH) \cap BH} \left| \left(\frac{\rho(t^{-1}x)}{\rho(x)} \right)^{1/2} \langle \eta_1(t^{-1}x), \eta_2(x) \rangle_{\mathcal{K}} d\mu_\rho(xH) \right| \\
&\leq \sum_{i,j=1}^{N_1, N_2} \int_{t(h_i KH) \cap h'_j KH} \left| \left(\frac{\rho(t^{-1}x)}{\rho(x)} \right)^{1/2} \langle \eta_1(t^{-1}x), \eta_2(x) \rangle_{\mathcal{K}} d\mu_\rho(xH) \right|.
\end{aligned} \tag{4}$$

To obtain the last inequality from (2) and the constant C we use the following remark. Recall that $\eta_1, \eta_2 \in \text{span}(\{\xi_{f,v} \mid f \in C_c(G), v \in \mathcal{K}\})$. We claim that the scalar product $|\langle \eta_1(t^{-1}x), \eta_2(x) \rangle_{\mathcal{K}}|$ is a bounded function in $x \in G$ and this upper-bound does not depend on t or on the domains $\{t(h_i KH) \cap h'_j KH\}_{h_i, h'_j}$. Indeed, for simplicity, consider that $\eta_1 = \xi_{f_1, v_1}$ and $\eta_2 = \xi_{f_2, v_2}$, where $f_1, f_2 \in C_c(G)$ and $v_1, v_2 \in \mathcal{K}$. In this case we have that:

$$\begin{aligned}
& |\langle \xi_{f_1, v_1}(t^{-1}x), \xi_{f_2, v_2}(x) \rangle_{\mathcal{K}}| \\
&= \left| \int_H \int_H \langle f_1(t^{-1}xh_1)\sigma(h_1)(v_1), f_2(xh_2)\sigma(h_2)(v_2) \rangle_{\mathcal{K}} dh_1 dh_2 \right| \\
&\leq \int_H \int_H |\langle f_1(t^{-1}xh_1)\sigma(h_1)(v_1), f_2(xh_2)\sigma(h_2)(v_2) \rangle_{\mathcal{K}}| dh_1 dh_2 \\
&\leq \int_H \int_H |f_1(t^{-1}xh_1)| \cdot |f_2(xh_2)| \cdot \|v_1\|_{\mathcal{K}} \cdot \|v_2\|_{\mathcal{K}} dh_1 dh_2 < C,
\end{aligned}$$

where C is a constant which does not depend on t . From here the conclusion follows. \square

Remark 3.9. Lemma 3.8 can be used in the following way. In order to study matrix coefficients of induced unitary representations, it is enough to evaluate integrals of the form $\int_{t_n(f_1 KH) \cap f_2 KH} \left(\frac{\rho(t_n^{-1}x)}{\rho(x)} \right)^{1/2} d\mu_\rho(xH)$, where f_1, f_2 are considered to be fixed and $t_n \rightarrow \infty$.

4 Vanishing results for the universal group $U(F)^+$

In this section we consider parabolically induced unitary representations of the universal group \mathbb{G} . We split this study in two cases: the unimodular case, when \mathbb{G}_ξ does not

contain hyperbolic elements, and the general case, when \mathbb{G}_ξ does contain hyperbolic elements. The following lemma is an easy but useful fact in the sequel.

Lemma 4.1. *Let F be primitive and let H be a closed, non-compact and proper subgroup of \mathbb{G} . Then, for every $x \in \mathcal{T}$, H_x does not have finite index in \mathbb{G}_x .*

Proof. By Caprace–De Medts [CDM11, Prop. 4.1] (see Proposition 2.4), notice that H cannot be an open subgroup of \mathbb{G} . Suppose there exists $x \in \mathcal{T}$ such that $[\mathbb{G}_x : H_x] < \infty$. As H_x is closed in \mathbb{G}_x and of finite index, we have that H_x is open in \mathbb{G}_x and so also in \mathbb{G} . This means that H is open in \mathbb{G} , obtaining thus a contradiction with the hypothesis. \square

4.1 The unimodular case

The name of this subsection is coming from the following remark.

Remark 4.2. Let $\xi \in \partial\mathcal{T}$. If H is a closed subgroup of \mathbb{G}_ξ that does not contain hyperbolic elements then H is unimodular. This is because H can be written as a countable union of compact subgroups.

Lemma 4.3. *Let F be primitive, $x \in \mathcal{T}$ and $\xi \in \partial\mathcal{T}$. Set $K := \mathbb{G}_x$ and let H be a closed, non-compact subgroup of \mathbb{G}_ξ , not containing hyperbolic elements.*

Let $g \in \mathbb{G}$. If $gKH \cap KH \neq \emptyset$, then there exists $k_g \in K$ such that

$$gKH \cap KH \subset k_g \mathbb{G}_{[x, x_g]} H,$$

where $x_g \in [x, \xi)$ with the properties that $\text{dist}_{\mathcal{T}}(x, x_g) = \frac{\text{dist}_{\mathcal{T}}(x, g(x))}{2}$ and k_g sends $[x, x_g]$ into the first half of the geodesic segment $[x, g(x)]$.

Proof. To prove the lemma, we have to evaluate the intersection $gKH \cap KH$. Suppose that $gKH \cap KH \neq \emptyset$. We want to determine the domain in K of the variable k' such that $g = k'hk$, with $h \in H$ and $k' \in K$.

Notice that, from the above equation $g = k'hk$, we have that:

$$\text{dist}_{\mathcal{T}}(x, g(x)) = \text{dist}_{\mathcal{T}}(x, k'h(x)) = \text{dist}_{\mathcal{T}}(x, h(x)). \quad (5)$$

As h is not hyperbolic, denote by x_h the first vertex of the geodesic ray $[x, \xi)$ fixed by h . We claim that x_h is a precise point on the geodesic ray $[x, \xi)$ determined only by the element g and not by the non-hyperbolic element h .

Indeed, from the equation (5) we must have that the vertex x_h is the midpoint of the geodesic segment $[x, h(x)]$. We obtain that $\text{dist}_{\mathcal{T}}(x, x_h) = \frac{\text{dist}_{\mathcal{T}}(x, g(x))}{2}$. Our claim follows and we set x_h simply by x_g .

Because $k'([x, h(x)]) = [x, g(x)]$, we have that k' sends the geodesic segment $[x, x_g]$ into the first half of the geodesic segment $[x, g(x)]$. We conclude that $k' \in k_g \mathbb{G}_{[x, x_g]}$, where $k_g \in K$ is a fixed element sending $[x, x_g]$ into the first half of the geodesic segment $[x, g(x)]$. \square

Theorem 4.4. *Let F be primitive and let $\xi \in \partial \mathcal{T}$. Let H be a closed, non-compact subgroup of \mathbb{G}_ξ , not containing hyperbolic elements. Let (σ, \mathcal{K}) be a unitary representation of H and consider on \mathbb{G}/H a \mathbb{G} -quasi-invariant regular Borel measure μ_ρ given by the rho-function $\rho : \mathbb{G} \rightarrow \mathbb{R}_+^*$.*

Then the induced unitary representation $(\pi_{\sigma, \mu_\rho}, \mathcal{H}_{\mu_\rho})$ of \mathbb{G} has all its matrix coefficients vanishing at infinity.

Proof. Fix for what follows a vertex $x \in \mathcal{T}$ and set $K := \mathbb{G}_x$.

By Remark 3.5 and because H, \mathbb{G} are unimodular, it is enough to consider the case when the rho-function ρ is the constant function $\mathbb{1}$ on \mathbb{G} . In this particular case, the measure $\mu_{\mathbb{1}}$ on \mathbb{G}/H is \mathbb{G} -invariant.

Assume there exist a sequence $\{t_n\}_{n>0}$ of \mathbb{G} and $\eta_1, \eta_2 \in \mathcal{H}_{\mu_{\mathbb{1}}}$ such that $t_n \rightarrow \infty$ and $|\langle \pi_{\sigma, \mu_{\mathbb{1}}}(t_n)\eta_1, \eta_2 \rangle| \not\rightarrow 0$. To the sequence $\{t_n\}_{n>0}$ apply Lemma 3.7 and then Lemma 3.8. Moreover, by Remark 3.9 it is enough to evaluate $\mu_{\mathbb{1}}(t_n(h_iKH) \cap h'_jKH)$, where h_i, h'_j are considered to be fixed and $t_n \rightarrow \infty$.

Notice that $\mu_{\mathbb{1}}(t_n(h_iKH) \cap h'_jKH) = \mu_{\mathbb{1}}((h'_j)^{-1}t_nh_iKH \cap KH)$. Apply then Lemma 4.3 to $g_n := (h'_j)^{-1}t_nh_i$. We obtain: $g_nKH \cap KH \subset k_{g_n}\mathbb{G}_{[x, x_{g_n}]}H$, where $x_{g_n} \in [x, \xi)$ with one of the properties being that $\text{dist}_{\mathcal{T}}(x, x_{g_n}) = \frac{\text{dist}_{\mathcal{T}}(x, g_n(x))}{2}$. As $t_n \rightarrow \infty$, we also have that $g_n \rightarrow \infty$; in addition, $\text{dist}_{\mathcal{T}}(x, x_{g_n}) \rightarrow \infty$ when $n \rightarrow \infty$.

Therefore, to evaluate $\mu_{\mathbb{1}}(g_nKH \cap KH)$ it is enough to calculate $\mu_{\mathbb{1}}(k_{g_n}\mathbb{G}_{[x, x_{g_n}]}H) = \mu_{\mathbb{1}}(\mathbb{G}_{[x, x_{g_n}]}H)$, where $\text{dist}_{\mathcal{T}}(x, x_{g_n}) \xrightarrow{g_n \rightarrow \infty} \infty$.

We claim that $\mu_{\mathbb{1}}(\mathbb{G}_{[x, x_{g_n}]}H) \xrightarrow{g_n \rightarrow \infty} 0$. Indeed, first notice that the index in K of $\mathbb{G}_{[x, x_{g_n}]}$ is finite, for every g_n . Moreover, $[K : \mathbb{G}_{[x, x_{g_n}]}] \xrightarrow{g_n \rightarrow \infty} \infty$. If this was not the case, we would have that $[K : \mathbb{G}_{[x, x_{g_n}]}] < \text{const}$ when $g_n \rightarrow \infty$. This would imply that $[K : \mathbb{G}_{[x, \xi]}] < \infty$, which is a contradiction with Lemma 4.1 applied to \mathbb{G}_ξ . As $\mu_{\mathbb{1}}(KH) < \infty$, $\mu_{\mathbb{1}}$ is \mathbb{G} -invariant, and so K -invariant, our claim follows easily.

We obtained that $\mu_{\mathbb{1}}(g_nKH \cap KH) \xrightarrow{g_n \rightarrow \infty} 0$, which is a contradiction with our assumption that $|\langle \pi_{\sigma, \mu_{\mathbb{1}}}(t_n)\eta_1, \eta_2 \rangle| \not\rightarrow 0$, when $t_n \rightarrow \infty$. The theorem stands proven. \square

4.2 The general case

Let F be primitive and let $\xi \in \partial \mathcal{T}$. In this subsection we consider that H is a closed subgroup of \mathbb{G}_ξ that does contain hyperbolic elements. This implies that H is not compact.

4.2.1 Structure and modular function of parabolic subgroups

Lemma 4.5. *Let F be primitive and let $\xi \in \partial \mathcal{T}$ be such that \mathbb{G}_ξ contains hyperbolic elements. Let $H \leq \mathbb{G}_\xi$ be a closed subgroup that also contains hyperbolic elements.*

Then there exists a hyperbolic element $\gamma \in H$, of attracting endpoint ξ , that is minimal, in the sense that any other hyperbolic element $\gamma' \in H$ is written $\gamma' = \gamma^n h$, where $n \in \mathbb{Z}$, $|n||\gamma| = |\gamma'|$ and $h \in H \cap \mathbb{G}_\xi^0$.

Proof. Let $Hyp(H) := \{\gamma \in H \mid \gamma \text{ is hyperbolic}\}$. Let $hyp_H := \min_{\gamma \in Hyp(H)} (|\gamma|)$. Notice that hyp_H exists. As in \mathbb{G} all hyperbolic elements have even translation length, we have that hyp_H is even and $hyp_H \geq 2$.

Let fix $\gamma \in H$ such that $|\gamma| = hyp_H$. Fix also a vertex x in $\text{Min}(\gamma)$. Moreover, consider that the attracting endpoint of γ is ξ .

Let $\gamma' \in Hyp(H)$ and let $x_{\gamma'}$ be the first vertex of $[x, \xi)$ contained in $\text{Min}(\gamma')$. Moreover, it is easy to see that $|\gamma'|$ is a multiple of $|\gamma|$, as otherwise, γ would not be of minimal translation length in H . Assume firstly that the attracting endpoint of γ' is ξ . Then $\gamma^{-n}\gamma'(x_{\gamma'}) = x_{\gamma'}$, where $n|\gamma| = |\gamma'|$. Thus, $\gamma' = \gamma^n h$, where $h \in H_{[x_{\gamma'}, \xi)}$. If γ' has ξ as a repelling endpoint, then $\gamma^n \gamma'((\gamma')^{-1}(x_{\gamma'})) = (\gamma')^{-1}(x_{\gamma'})$, where $n|\gamma| = |\gamma'|$. Thus $\gamma' = \gamma^{-n} h$, where now h is in $H_{[(\gamma')^{-1}(x_{\gamma'}), \xi)}$. \square

The aim of the next lemma is to evaluate the modular function of a closed, non-compact subgroup of \mathbb{G}_ξ , where $\xi \in \partial \mathcal{T}$, and which does contain hyperbolic elements.

Lemma 4.6. *Let F be primitive and let $\xi \in \partial \mathcal{T}$ be such that \mathbb{G}_ξ contains hyperbolic elements. Let $H \leq \mathbb{G}_\xi$ be a closed subgroup that also contains hyperbolic elements. Let γ be a minimal hyperbolic element of H given by Lemma 4.5, with attracting endpoint ξ . Let x be a vertex of $\text{Min}(\gamma)$.*

Then $\frac{1}{(d-1)^{|\gamma|}} \leq \Delta_H(\gamma) = \frac{1}{[H_{[\gamma(x), \xi)} : H_{[x, \xi)}]} < 1$.

Proof. Notice that for every $h \in H \cap \mathbb{G}_\xi^0$, $\Delta_H(g) = 1$. This is because $H \cap \mathbb{G}_\xi^0$ is a countable union of compact subgroups.

By Lemma 4.5, it is sufficient to evaluate the modular function of H only for the element γ . By convention, the attracting endpoint of γ is ξ . Notice the following facts. Firstly, $H_{[x, \xi)} \leq H_{[\gamma(x), \xi)}$ and secondly, the index $[H_{[\gamma(x), \xi)} : H_{[x, \xi)}] \leq (d-1)^{|\gamma|}$, where d is the regularity of the tree \mathcal{T} . Moreover, $H_{[\gamma(x), \xi)} = \gamma H_{[x, \xi)} \gamma^{-1}$. Indeed, let $h \in H_{[x, \xi)}$. Then, $\gamma h \gamma^{-1}(\gamma(x)) = \gamma(x)$. Thus, $\gamma H_{[x, \xi)} \gamma^{-1} \leq H_{[\gamma(x), \xi)}$. Let now $h \in H_{[\gamma(x), \xi)}$. Then, $h(\gamma(x)) = \gamma(x)$, so $\gamma^{-1} h \gamma \in H_{[x, \xi)}$. We obtained that $H_{[\gamma(x), \xi)} \leq \gamma H_{[x, \xi)} \gamma^{-1}$ and also the aimed equality.

Let dh denote the left Haar measure on H . Then

$$dh(H_{[\gamma(x), \xi)}) = dh(\gamma H_{[x, \xi)} \gamma^{-1}) = dh(H_{[x, \xi)} \gamma^{-1}) = \Delta_H(\gamma^{-1}) dh(H_{[x, \xi)}).$$

As the Haar measure is left and right invariant with respect to $H \cap \mathbb{G}_\xi^0$, we have that

$$dh(H_{[\gamma(x), \xi)}) = dh(H_{[x, \xi)}) \cdot [H_{[\gamma(x), \xi)} : H_{[x, \xi)}].$$

From the above two equalities we obtain

$$1 < \Delta_H(\gamma^{-1}) = [H_{[\gamma(x), \xi)} : H_{[x, \xi)}] \leq (d-1)^{|\gamma|}.$$

\square

4.2.2 The main idea

In order to prove that parabolically induced unitary representations of the universal group \mathbb{G} have all their matrix coefficients vanishing at infinity, we make use of Remark 3.9. Therefore, the next lemmas evaluate integrals of the form

$$\int_{gf_1KH \cap f_2KH} \left(\frac{\rho(g^{-1}x)}{\rho(x)} \right)^{1/2} d\mu_\rho(xH),$$

where $g, f_1, f_2 \in \mathbb{G}$.

Lemma 4.7. *Let F be primitive, x be a vertex in \mathcal{T} and $\xi \in \partial\mathcal{T}$. Set $K := \mathbb{G}_x$ and let H be a subgroup of \mathbb{G} .*

Let $g, f_1, f_2 \in \mathbb{G}$. Then $gf_1KH \cap f_2KH = \sqcup_{i \in I} f_2k_iH$, where $\{k_i\}_{i \in I} \subset K/(K \cap H)$ are pairwise disjoint. In addition, for every $i \in I$, there is a unique $k_i \in K/(K \cap H)$ and a unique $h_i \in H$ such that $gf_1k_i = f_2k_ih_i \in f_2k_iH$.

Proof. Let $x \in gf_1KH \cap f_2KH$. Then there exist $k, k' \in K$ and $h, h' \in H$, such that $x = gf_1kh = f_2k'h'$; so $xh^{-1} = gf_1k = f_2k'h'h^{-1}$. By taking $k' \in K/(K \cap H)$ we obtain the first part of the lemma. Suppose there are $k, k' \in K/(K \cap H)$ and $h, h' \in H$ such that $k \notin k'H$ and $gf_1k = f_2k_ih, gf_1k' = f_2k_ih' \in f_2k_iH$. From here is easy to see that in fact $k = k'$ and $h = h'$. The lemma is proved. \square

Lemma 4.8. *Let F be primitive and let $\xi \in \partial\mathcal{T}$. Let H be a closed subgroup of \mathbb{G}_ξ that contains hyperbolic elements. Let γ be a minimal hyperbolic element of H given by Lemma 4.5, with attracting endpoint ξ , and let x be a vertex in $\text{Min}(\gamma)$. Set $K := \mathbb{G}_x$. Consider on \mathbb{G}/H a \mathbb{G} -quasi-invariant regular Borel measure μ_ρ given by the rho-function $\rho : \mathbb{G} \rightarrow \mathbb{R}_+^*$.*

Let $g, f_1, f_2 \in \mathbb{G}$. Then there is a constant $C > 0$, depending only on K, ρ and f_1, f_2 such that

$$\int_{gf_1KH \cap f_2KH} \left(\frac{\rho(g^{-1}x)}{\rho(x)} \right)^{1/2} d\mu_\rho(xH) \leq C \int_{\sqcup_{i \in I} f_2k_iH} \Delta_H(h_i)^{-1/2} d\mu_\rho(f_2k_iH),$$

where I, k_i, h_i are given by Lemma 4.7.

Proof. By Lemma 4.7, we have that $gf_1KH \cap f_2KH = \sqcup_{i \in I} f_2k_iH$, where $\{k_i\}_{i \in I} \subset K/(K \cap H)$ are pairwise disjoint. Let $x \in gf_1KH \cap f_2KH$. Then, by the same Lemma 4.7, $x = gf_1k_ih = f_2k_ih_ih$, for some $h \in H$ and some $i \in I$. Therefore, $\left(\frac{\rho(g^{-1}x)}{\rho(x)} \right)^{1/2} = \left(\frac{\rho(f_1k_ih)}{\rho(f_2k_ih_ih)} \right)^{1/2} = \left(\frac{\rho(f_1k_i)\Delta_H(h)}{\rho(f_2k_i)\Delta_H(h_ih)} \right)^{1/2}$. As the map ρ is continuous on \mathbb{G} and K is compact, there exists a constant $C > 0$ such that $0 \leq \left(\frac{\rho(f_1k)}{\rho(f_2k')} \right)^{1/2} \leq C$, for every $k, k' \in K$. We obtain that $\left(\frac{\rho(g^{-1}x)}{\rho(x)} \right)^{1/2} \leq C \Delta_H(h_i)^{-1/2}$, for $x \in f_2k_iH$. The conclusion follows. \square

4.2.3 A key lemma

As it is proved in Section 4.2.2, it remains to integrate the modular function Δ_H of H on the intersection $gf_1KH \cap f_2KH = \sqcup_{i \in I} f_2k_iH$, for $g, f_1, f_2 \in \mathbb{G}$. In order to do that, we need to investigate more closely the set $\{h_i\}_{i \in I}$, given by Lemma 4.7. Even if h_i is uniquely determined by k_i , for every $i \in I$, we still can have that two h_i, h_j , with $i \neq j \in I$, can belong to the same right coset of $H/(H \cap \mathbb{G}_\xi^0)$, thus $\Delta_H(h_i) = \Delta_H(h_j)$. Otherwise saying, for a given right coset $[h] \in H/(H \cap \mathbb{G}_\xi^0)$, the equation $k_1f_2^{-1}gf_1k_2 = h$, with $k_1, k_2 \in K/(K \cap H)$ and $h \in [h]$ can have many solutions; denote by $K_{[h]} \subset K$ the domain of solutions for k_1 . Consequently, to bound from above the integral $\int_{\sqcup_{i \in I} f_2k_iH} \Delta_H(h_i)^{-1/2} d\mu_\rho(f_2k_iH)$, obtained by Lemma 4.8, it is enough to evaluate the set of all right cosets $[h_i] \in H/(H \cap \mathbb{G}_\xi^0)$, for $i \in I$, and for each such right coset $[h]$ to evaluate $\mu_\rho(K_{[h]})\Delta_H(h)^{-1/2}$. The next lemma is the key ingredient to prove that ‘summing up’ $\mu_\rho(K_{[h]})\Delta_H(h)^{-1/2}$, we obtain a sequence that tends to zero, when $g \rightarrow \infty$.

Lemma 4.9. *Let $p > 0 \in \mathbb{N}$ and let $\frac{1}{(d-1)^{2p}} \leq t < 1$, where d is the regularity of \mathcal{T} . Consider a sequence $\{M_n\}_{n>0} \subset \mathbb{N}$ with the following properties:*

- 1) *for every $n > 0$, $M_n = 2p \cdot m_n + 2r_n$, where $0 \leq 2r_n < 2p$*
- 2) *$m_n \xrightarrow{n \rightarrow \infty} \infty$.*

Consider the sequence $\{S_n\}_{n>0} \subset \mathbb{R}_+$, defined by

$$S_n := \sum_{k=0}^{\lfloor m_n/2 \rfloor} \frac{1}{d(d-1)^{r_n+k2p-1}} t^{(m_n-2k)/2} + \frac{1}{d(d-1)^{M_n/2-1}} + \sum_{j=1}^{m_n} \frac{1}{d(d-1)^{(M_n+j2p)/2-1}} t^{-j/2},$$

where by convention $\frac{1}{d(d-1)^{r_n+k2p-1}} = 1$ when $r_n = k = 0$.

Then $S_n \rightarrow 0$, when $n \rightarrow \infty$.

Proof. Notice that $M_n \xrightarrow{n \rightarrow \infty} \infty$, so $\frac{1}{d(d-1)^{M_n/2-1}} \xrightarrow{n \rightarrow \infty} 0$. For the last sum in S_n we have

$$\sum_{j=1}^{m_n} \frac{1}{d(d-1)^{(M_n+j2p)/2-1}} t^{-j/2} \leq \sum_{j=1}^{m_n} \frac{1}{d(d-1)^{(M_n+j2p)/2-1}} (d-1)^{jp} = \frac{m_n}{d(d-1)^{M_n/2-1}}.$$

One has that $\frac{m_n}{d(d-1)^{M_n/2-1}} \xrightarrow{n \rightarrow \infty} 0$.

Consider now the first summand in S_n . If m_n is even, then

$$\begin{aligned} \sum_{k=0}^{\lfloor m_n/2 \rfloor} \frac{1}{d(d-1)^{r_n+k2p-1}} t^{(m_n-2k)/2} &= \sum_{k=0}^{m_n/2-1} \frac{1}{d(d-1)^{r_n+k2p-1}} t^{(m_n-2k)/2} + \frac{1}{d(d-1)^{M_n/2-1}} \\ &< \sum_{k=0}^{m_n/2-1} \frac{1}{(d-1)^{k2p}} t^{(m_n-2k)/2} + \frac{1}{d(d-1)^{M_n/2-1}}. \end{aligned}$$

As $\frac{1}{d(d-1)^{M_n/2-1}} \xrightarrow{n \rightarrow \infty} 0$, it remains to evaluate $\sum_{k=0}^{m_n/2-1} \frac{1}{(d-1)^{k2p}} t^{(m_n-2k)/2}$. We have

$$\begin{aligned} \sum_{k=0}^{m_n/2-1} \frac{1}{(d-1)^{k2p}} t^{(m_n-2k)/2} &< \sum_{k=0}^{\lfloor (m_n/2-1)/2 \rfloor} t^{(m_n-2k)/2} \\ &+ \sum_{k=\lfloor (m_n/2-1)/2 \rfloor + 1}^{m_n/2-1} \frac{1}{(d-1)^{k2p}} t^{(m_n-2k)/2} \\ &< \sum_{k=0}^{\lfloor (m_n/2-1)/2 \rfloor} t^{(m_n-2k)/2} \\ &+ \frac{1}{(d-1)^{(\lfloor (m_n/2-1)/2 \rfloor + 1)2p}} \sum_{k=\lfloor (m_n/2-1)/2 \rfloor + 1}^{m_n/2-1} t^{(m_n-2k)/2}. \end{aligned}$$

As $m_n \xrightarrow{n \rightarrow \infty} 0$, we have that $\frac{1}{(d-1)^{(\lfloor (m_n/2-1)/2 \rfloor + 1)2p}} \xrightarrow{n \rightarrow \infty} 0$ and we obtain

$$\frac{1}{(d-1)^{(\lfloor (m_n/2-1)/2 \rfloor + 1)2p}} \sum_{k=\lfloor (m_n/2-1)/2 \rfloor + 1}^{m_n/2-1} t^{(m_n-2k)/2} \xrightarrow{n \rightarrow \infty} 0.$$

Moreover, $\sum_{k=0}^{\lfloor (m_n/2-1)/2 \rfloor} t^{(m_n-2k)/2} \xrightarrow{m_n \rightarrow \infty} 0$, because $(m_n - 2\lfloor (m_n/2-1)/2 \rfloor)/2 \xrightarrow{n \rightarrow \infty} \infty$.

If m_n is odd, we proceed in the same way. In this case the summand $\frac{1}{d(d-1)^{M_n/2-1}}$ does not appear. After these evaluations, we conclude that indeed $S_n \rightarrow 0$, when $n \rightarrow \infty$. \square

4.2.4 Evaluation of $gKH \cap KH$

The last key step to prove that parabolically induced unitary representations of the universal group \mathbb{G} are C_0 , is the evaluation of $gKH \cap KH$, when $g \in A^+$. This is given by the next technical proposition. First we need a definition.

Definition 4.10. Let x be a vertex of \mathcal{T} and ξ be an endpoint of $\partial\mathcal{T}$. The set A^+ is considered with respect to the edge in the star of x that belongs to the geodesic ray $[x, \xi]$. Denote this edge by e . Set $K := \mathbb{G}_x$. Define the map $\text{proj}_{(x, \xi]} : A^+ \rightarrow (x, \xi]$ by $\text{proj}_{(x, \xi]}(g)$ is the vertex or the endpoint ξ with the property that $[x, \xi_{g,+}] \cap [x, \xi] = [x, \text{proj}_{(x, \xi]}(g)]$, where $\xi_{g,+}$ is the attracting endpoint of g . As $g \in A^+$, notice that $\text{proj}_{(x, \xi]}(g)$ is indeed a point in $(x, \xi]$. Let now $g \in \mathbb{G}$ be a hyperbolic element that translates the vertex x . Consider its K -double coset KgK and set $\text{proj}_{(x, \xi]}(KgK) := \max_{g' \in A^+ \cap KgK} \{\text{proj}_{(x, \xi]}(g')\}$.

Proposition 4.11. Let F be primitive and let $\xi \in \partial\mathcal{T}$. Let H be a closed subgroup of \mathbb{G}_ξ that contains hyperbolic elements. Let γ be a minimal hyperbolic element of H given

by Lemma 4.5, with attracting endpoint ξ , and let x be a vertex of $\text{Min}(\gamma)$. Set $K := \mathbb{G}_x$ and let A^+ such that $\gamma \in A^+$.

Let $g \in A^+$. Assume that $\text{proj}_{(x,\xi]}(KgK) = \text{proj}_{(x,\xi]}(g)$. Assume also there exist $k_1, k_2 \in K \setminus \{H \cap K\}$ and $h \in H$ such that $k_1 g k_2 = h = \gamma^n h_0$, where $h_0 \in H \cap \mathbb{G}_\xi^0$ and $n \in \mathbb{Z}$.

Then $0 \leq |n| \leq \frac{\text{dist}_{\mathcal{T}}(x, g(x))}{|\gamma|}$ and $k_1 \in \mathbb{G}_{[x, x_h]}$, where $x_h \in [x, \text{proj}_{(x,\xi]}(g)]$ is such that $\text{dist}_{\mathcal{T}}(x, x_h) = \frac{\text{dist}_{\mathcal{T}}(x, g(x)) + \text{sign}(n)|\gamma^n|}{2}$, where $\text{sign}(0) = 0$.

Proof. Let ξ_+ and ξ_- be the attracting and the repelling endpoints of g . As by hypothesis k_2 is not fixing ξ , we denote by x_{k_2} the vertex of the geodesic line (ξ_-, ξ) with the property that $[x, k_2(\xi)) \cap (\xi_-, \xi) = [x, x_{k_2}]$. We have three cases: either $x_{k_2} \in [x, \xi)$ or $x_{k_2} \in (x, g^{-1}(x))$ or $x_{k_2} \in [g^{-1}(x), \xi_-)$.

Suppose that $x_{k_2} \in [x, \xi)$. Because $k_1 g k_2(\xi) = \xi$, $k_1 g k_2(e)$ is an edge of $\mathcal{T}_{x,e}$ and the orientation of $k_1 g k_2(e)$ induced from e points towards the boundary $\partial T_{x,e}$, like e . Therefore, we have that $k_1 g k_2 \in A^+$. In addition, we know that $k_1 g k_2 \in H$, thus $h = k_1 g k_2 \in A^+ \cap H$. As by hypothesis g is such that $\text{proj}_{(x,\xi]}(KgK) = \text{proj}_{(x,\xi]}(g)$, we conclude that $g \in A^+ \cap H$. In addition, by Lemma 4.5 we have that $h = \gamma^n h_0$, where $h_0 \in K \cap H$; thus $|g| = |h| = n|\gamma|$. As $k_1 g k_2(\xi) = \xi$ and because g is hyperbolic, with attracting endpoint ξ and with $x \in \text{Min}(g)$, we obtain that k_1 must fix at least the vertex $g(x) \in (x, \xi)$. Therefore, $k_1 \in \mathbb{G}_{[x, x_h]}$, where $x_h = g(x)$. In this case, the conclusion of the proposition is proven.

Suppose that $x_{k_2} \in [g^{-1}(x), \xi_-)$. Then $g k_2(e)$ is an edge of $\mathcal{T}_{x,e}$ and the orientation of $g k_2(e)$ induced from e points outwards the boundary $\partial T_{x,e}$, thus towards e . Because $x_{k_2} \in [g^{-1}(x), \xi_-)$, by applying k_1 to $g k_2$, we obtain that $k_1(\mathcal{T}_{x,e}) \cap \mathcal{T}_{x,e} = \{x\}$ and the edge $k_1 g k_2(e)$ points towards the edge e . Therefore $k_1 g k_2$ must be a hyperbolic element (of H) translating the vertex x outwards the half-tree $\mathcal{T}_{x,e}$. Consequently, we have that ξ is the repelling endpoint of $k_1 g k_2$. Otherwise saying, ξ is the attracting endpoint of the hyperbolic element $(k_1 g k_2)^{-1} = h^{-1} \in H$ and $x \in \text{Min}(h^{-1})$. We have that $|h| = |h^{-1}| = \text{dist}_{\mathcal{T}}(x, g(x)) = |n||\gamma|$. Although we can say more, we do not impose any restriction for k_1 , so $k_1 \in \mathbb{G}_{[x, x_h]}$ where $x_h = x$. The conclusion of the proposition is still valid in this case.

Suppose now that $x_{k_2} \in (g^{-1}(x), x)$. We claim that in this case we have $g(x_{k_2}) \in [x, \text{proj}_{(x,\xi]}(g)]$. Indeed, supposing the contrary we have that $\text{proj}_{(x,\xi]}(g) \in (x, g(x_{k_2}))$. Then $g(x_{k_2}) \notin [x, \xi)$. As the geodesic ray $[x_{k_2}, k_2(\xi))$ is sent by g into the geodesic ray $[g(x_{k_2}), g k_2(\xi))$ we would notice that $[g(x_{k_2}), g k_2(\xi))$ does not intersect $[x, \xi)$. However, by applying k_1 , we must have that $k_1 g(x_{k_2}) \in [x, \xi)$, as $k_1 g k_2(\xi) = \xi$. This is a contradiction with our hypothesis that $\text{proj}_{(x,\xi]}(KgK) = \text{proj}_{(x,\xi]}(g)$ and the claim follows. As $k_1 g k_2(\xi) = \xi$, from the latter claim we immediately have that $k_1 \in \mathbb{G}_{[x, g(x_{k_2})]}$. From here we deduce the following two facts:

- 1) the segment $[x, k_2^{-1}(x_{k_2}))$, where $k_2^{-1}(x_{k_2}) \in (x, \xi)$, is sent by $h = k_1 g k_2$ into the segment $(g(x_{k_2}), k_1 g(x)) \subset \mathcal{T}_{x,e} \setminus \{[x, \xi)\}$, and the orientation is reversed

- 2) the edge $k_1 g k_2(e)$ belongs to $\mathcal{T}_{x,e}$ and the orientation of $k_1 g k_2(e)$ induced from e would point outwards the boundary $\partial T_{x,e}$, thus towards e . Therefore, either $k_1 g k_2$ is elliptic, or $k_1 g k_2$ is hyperbolic in H , with translation length strictly smaller than $\text{dist}_{\mathcal{T}}(x, g(x))$.

Our next claim is that h is elliptic if and only if $\text{dist}_{\mathcal{T}}(x, x_{k_2}) = \frac{\text{dist}_{\mathcal{T}}(x, g^{-1}(x))}{2}$. Suppose that $h = k_1 g k_2$ is elliptic. Then by the above fact 1) we know that the segment $[x, k_2^{-1}(x_{k_2})]$ does not intersect $[x, \xi]$ by applying h . Therefore, as $h \in H$ is elliptic, we have that h fixes the midpoint of the segment $[k_2^{-1}(x_{k_2}), h(k_2^{-1}(x_{k_2}))] = [k_2^{-1}(x_{k_2}), g(x_{k_2})]$. We deduce that in fact $k_2^{-1}(x_{k_2}) = h(k_2^{-1}(x_{k_2})) = g(x_{k_2})$, from where it is easy to see that $\text{dist}_{\mathcal{T}}(x, x_{k_2}) = \frac{\text{dist}_{\mathcal{T}}(x, g^{-1}(x))}{2}$. Suppose now that $\text{dist}_{\mathcal{T}}(x, x_{k_2}) = \frac{\text{dist}_{\mathcal{T}}(x, g^{-1}(x))}{2}$. Then, we have that $k_2^{-1}(x_{k_2}) = g(x_{k_2})$. As $k_1 \in \mathbb{G}_{[x, g(x_{k_2})]}$, we conclude that $h(k_2^{-1}(x_{k_2})) = k_1 g k_2(k_2^{-1}(x_{k_2})) = g(x_{k_2}) = k_2^{-1}(x_{k_2})$, so h is elliptic. The equivalence follows.

For h elliptic, we resume the following: $k_1 g k_2 \in H \cap \mathbb{G}_{\xi}^0$, so $n = 0$, and $k_1 \in \mathbb{G}_{[x, x_h]}$, where $x_h = g(x_{k_2}) \in [x, \text{proj}_{(x, \xi]}(g)]$, with $\text{dist}_{\mathcal{T}}(x, g(x_{k_2})) = \frac{\text{dist}_{\mathcal{T}}(x, g(x))}{2}$. In this case the proposition is proved.

If $h = k_1 g k_2$ is hyperbolic, then $\text{dist}_{\mathcal{T}}(x, x_{k_2}) \neq \frac{\text{dist}_{\mathcal{T}}(x, g^{-1}(x))}{2}$.

Suppose that $\text{dist}_{\mathcal{T}}(x, x_{k_2}) < \frac{\text{dist}_{\mathcal{T}}(x, g^{-1}(x))}{2}$, this implies that

$$\frac{\text{dist}_{\mathcal{T}}(x, g(x))}{2} < \text{dist}_{\mathcal{T}}(x, g(x_{k_2})).$$

Moreover, using the above fact 1) and that h is hyperbolic fixing ξ , we conclude that ξ is the attracting endpoint of h and that h translates the vertex $k_2^{-1}(x_{k_2}) \in (x, \xi)$ to $k_1 g(x_{k_2}) = g(x_{k_2}) \in (x, \xi)$. By Lemma 4.5, we have that $h = \gamma^n h_0$, for some $h_0 \in H \cap \mathbb{G}_{\xi}^0$, and n is such that $n|\gamma| = |h| = \text{dist}_{\mathcal{T}}(x, g(x)) - 2 \text{dist}_{\mathcal{T}}(x, k_2^{-1}(x_{k_2})) < \text{dist}_{\mathcal{T}}(x, g(x))$. In addition, $k_1 \in \mathbb{G}_{[x, x_h]}$, where $x_h = g(x_{k_2}) \in [x, \text{proj}_{(x, \xi]}(g)]$ and indeed $\text{dist}_{\mathcal{T}}(x, x_h) = \frac{\text{dist}_{\mathcal{T}}(x, g(x)) - |\gamma^n|}{2} + |\gamma^n|$.

Suppose now that $\text{dist}_{\mathcal{T}}(x, x_{k_2}) > \frac{\text{dist}_{\mathcal{T}}(x, g^{-1}(x))}{2}$, this implies that

$$\frac{\text{dist}_{\mathcal{T}}(x, g(x))}{2} > \text{dist}_{\mathcal{T}}(x, g(x_{k_2})).$$

As before, using the above fact 1) and that h is hyperbolic fixing ξ , we conclude that ξ must be the repelling endpoint of h and that h^{-1} translates the vertex $g(x_{k_2}) \in \text{Min}(h) \cap (x, \xi)$ to $k_2^{-1}(x_{k_2}) \in (x, \xi)$. By Lemma 4.5, we have that $h = \gamma^{-n} h_0$, for some $h_0 \in H \cap \mathbb{G}_{\xi}^0$, and $n > 0$ is such that $n|\gamma| = |h| = \text{dist}_{\mathcal{T}}(x, g(x)) - 2 \text{dist}_{\mathcal{T}}(x, g(x_{k_2})) < \text{dist}_{\mathcal{T}}(x, g(x))$. In addition, $k_1 \in \mathbb{G}_{[x, x_h]}$, where $x_h = g(x_{k_2}) \in [x, \text{proj}_{(x, \xi]}(g)]$ and indeed $\text{dist}_{\mathcal{T}}(x, x_h) = \frac{\text{dist}_{\mathcal{T}}(x, g(x)) - |\gamma^{-n}|}{2}$.

□

4.2.5 The proof

We are now ready to give the proof when H contains hyperbolic elements.

Theorem 4.12. *Let F be primitive and let $\xi \in \partial \mathcal{T}$. Let H be a closed subgroup of \mathbb{G}_ξ , containing hyperbolic elements. Let (σ, \mathcal{K}) be a unitary representation of H and consider on \mathbb{G}/H a \mathbb{G} -quasi-invariant regular Borel measure μ_ρ given by the rho-function $\rho : \mathbb{G} \rightarrow \mathbb{R}_+^*$.*

Then the induced unitary representation $(\pi_{\sigma, \mu_\rho}, \mathcal{H}_{\mu_\rho})$ of \mathbb{G} has all its matrix coefficients vanishing at infinity.

Proof. By Lemma 4.5, let γ be a minimal hyperbolic element of H . Fix for what follows a vertex $x \in \text{Min}(\gamma)$ and set $K := \mathbb{G}_x$.

Assume there exist a sequence $\{t_n\}_{n>0}$ of \mathbb{G} and $\eta_1, \eta_2 \in \mathcal{H}_{\mu_\rho}$ such that $t_n \rightarrow \infty$ and $|\langle \pi_{\sigma, \mu_\rho}(t_n)\eta_1, \eta_2 \rangle| \rightarrow 0$. To the sequence $\{t_n\}_{n>0}$ apply Lemma 3.7 and then Lemma 3.8. By Remark 3.9 it is enough to evaluate integrals of the form

$$\int_{t_n(f_1KH) \cap f_2KH} \left(\frac{\rho(t_n^{-1}x)}{\rho(x)} \right)^{1/2} d\mu_\rho(xH),$$

where $f_1, f_2 \in \mathbb{G}$ are considered to be fixed and $t_n \rightarrow \infty$.

First of all, fix t_n . Apply Lemmas 4.7 and 4.8 to t_n, f_1, f_2 . One obtains

$$\int_{t_n(f_1KH) \cap f_2KH} \left(\frac{\rho(t_n^{-1}x)}{\rho(x)} \right)^{1/2} d\mu_\rho(xH) \leq C \int_{\sqcup_{i \in I_n} f_2k_{i,n}H} \Delta_H(h_{i,n})^{-1/2} d\mu_\rho(f_2k_{i,n}H),$$

where the constant $C > 0$ depends only on K, ρ, f_1, f_2 and the set $I_n, k_{i,n}, h_{i,n}$, with $i \in I_n$, depend on t_n, f_1 and f_2 .

As explained in the beginning of Section 4.2.3, we have to evaluate first the set of all right cosets $[h_{i,n}] \in H/(H \cap \mathbb{G}_\xi^0)$. This follows from Proposition 4.11. Indeed, for simplicity set $g_n := f_2^{-1}t_nf_1$. By Lemma 2.7, one can write $g_n = k\gamma_n k'$, where $k, k' \in K$ and $\gamma_n \in A^+$. This decomposition is unique up to the K -double coset Kg_nK . Therefore, we can choose γ_n such that $\text{proj}_{(x, \xi]}(Kg_nK) = \text{proj}_{(x, \xi]}(\gamma_n)$. Fix such γ_n, k, k' with $g_n = k\gamma_n k'$ and $\text{proj}_{(x, \xi]}(Kg_nK) = \text{proj}_{(x, \xi]}(\gamma_n)$. By Proposition 4.11, applied to γ_n , we have that, for every $i \in I_n$, $k_{i,n}^{-1}g_n k_{i,n} = k_{i,n}^{-1}k\gamma_n k'k_{i,n} = h_{i,n} = \gamma^{m_i}h_0$, with $0 \leq |m_i| \leq \frac{\text{dist}_{\mathcal{T}}(x, g_n(x))}{|\gamma|}$ and $h_0 \in H \cap \mathbb{G}_\xi^0$.

Evaluate now the solutions for the equation

$$k_1 g_n k_2 = k_1 k \gamma_n k' k_2 = h, \quad (6)$$

for a given right coset $[h] \in \{[h_{i,n}] \mid i \in I_n\} \subset H/(H \cap \mathbb{G}_\xi^0)$ and where $k_1 k, k' k_2 \in K/(K \cap H)$. Notice that for any element $h \in H$, satisfying equation (6), we have

$$\text{dist}_{\mathcal{T}}(x, h(x)) = \text{dist}_{\mathcal{T}}(x, g_n(x)) = \text{dist}_{\mathcal{T}}(x, \gamma_n(x)). \quad (7)$$

Apply again Proposition 4.11. We obtain that, for a given right coset $[h = \gamma^m] \in \{[h_{i,n}] \mid i \in I_n\} \subset H/(H \cap \mathbb{G}_\xi^0)$, $k_1^{-1} \in k\mathbb{G}_{[x,x_h]}$, where $x_h \in [x, \text{proj}_{(x,\xi)}(\gamma_n)]$ is such that $\text{dist}_{\mathcal{T}}(x, x_h) = \frac{\text{dist}_{\mathcal{T}}(x, g_n(x)) + \text{sign}(m)|\gamma^m|}{2}$. Notice that the index $[K : \mathbb{G}_{[x,x_h]}] \leq \frac{1}{d(d-1)^{\text{dist}_{\mathcal{T}}(x,x_h)-1}}$, where by convention $\frac{1}{d(d-1)^{\text{dist}_{\mathcal{T}}(x,x_h)-1}} = 1$, when $x = x_h$. Denote by $K_{\gamma^m} \subset K$ the domain of all solutions k_1^{-1} ; in addition $K_{\gamma^m} \subset k\mathbb{G}_{[x,x_h]}$. We have that

$$\mu_\rho(f_2 K_{\gamma^m} H) \Delta_H(\gamma)^{-m/2} \leq C_1 \mu_\rho(K_{\gamma^m} H) \Delta_H(\gamma)^{-m/2},$$

where we have applied Remark 3.6 for $K_{\gamma^m} \subset K$ and the constant C_1 depends on f_2 and K . However, by Lemma 3.3 and Remark 3.5 we can assume, without loss of generality, that μ_ρ is K -invariant.

Therefore

$$\begin{aligned} \mu_\rho(K_{\gamma^m} H) \Delta_H(\gamma)^{-m/2} &\leq \mu_\rho(k\mathbb{G}_{[x,x_h]} H) \Delta_H(\gamma)^{-m/2} = \mu_\rho(\mathbb{G}_{[x,x_h]} H) \Delta_H(\gamma)^{-m/2} \\ &= [K : \mathbb{G}_{[x,x_h]}] \Delta_H(\gamma)^{-m/2} \leq \frac{1}{d(d-1)^{\text{dist}_{\mathcal{T}}(x,x_h)-1}} \Delta_H(\gamma)^{-m/2}. \end{aligned}$$

It remains to apply Lemma 4.9 for $2p = |\gamma|$, $M_n = \text{dist}_{\mathcal{T}}(x, g_n(x))$ and $t = \Delta_H(\gamma)$. We obtain that

$$\int_{\sqcup_{i \in I_n} f_2 k_{i,n} H} \Delta_H(h_{i,n})^{-1/2} d\mu_\rho(f_2 k_{i,n} H) \xrightarrow[t_n \rightarrow \infty]{} 0,$$

which is a contradiction with our assumption that $|\langle \pi_{\sigma, \mu_\rho}(t_n) \eta_1, \eta_2 \rangle| \not\rightarrow 0$. The theorem stands proven. \square

4.3 The main Theorem

Theorems 4.4 and 4.12 give us the aimed result of this article:

Theorem 4.13. *Let F be primitive and let $\xi \in \partial \mathcal{T}$. Let H be a closed subgroup of \mathbb{G}_ξ . Let (σ, \mathcal{K}) be a unitary representation of H and consider on \mathbb{G}/H a \mathbb{G} -quasi-invariant regular Borel measure μ_ρ given by the rho-function $\rho : \mathbb{G} \rightarrow \mathbb{R}_+^*$.*

Then the induced unitary representation $(\pi_{\sigma, \mu_\rho}, \mathcal{H}_{\mu_\rho})$ of \mathbb{G} has all its matrix coefficients vanishing at infinity.

Proof. It remains to consider the case when H is a compact subgroup of \mathbb{G}_ξ . This is a particular case of the well-known general fact that all unitary representations of a locally compact subgroup that are induced from compact subgroups have all they matrix coefficients vanishing at infinity. For the idea of the proof the reader can consult the book of Bekka–de la Harpe–Valette [BdlHV08, Proposition C.4.6]. \square

Acknowledgements

We would like to thank Pierre-Emmanuel Caprace for addressing the question, in the beginning of the author's PhD thesis, if parabolically induced unitary representations

of the universal group $U(F)^+$, with F being primitive, have all their matrix coefficients vanishing at infinity. We thank Pierre-Emmanuel Caprace and Stefaan Vaes for pointing out a gap in an earlier version of this paper.

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